Function Spaces
Mid Semester Exam
Marks - 30, Duration - 2.15 Hours

1. [2 marks] Pick out the compact sets.
   (a) $\{(x,y) \in \mathbb{R}^2 : xy = 1\}$.
   (b) The set of all invertible matrices, in $M_n(\mathbb{R})$ is compact.
   (c) $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$.
   (d) The set of all nilpotent matrices, in $M_n(\mathbb{R})$ compact.

2. [2 marks] Pick out the true statements.
   (a) If $f : (0,1) \to \mathbb{R}$ is a continuous function, then f maps Cauchy sequences into Cauchy sequences.
   (b) If $f : [0,1] \to \mathbb{R}$ is a continuous function, then f maps Cauchy sequences into Cauchy sequences.
   (c) If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function, then f maps Cauchy sequences into Cauchy sequences.
   (d) If $f : \mathbb{R} \to \mathbb{R}$ is a uniform continuous function, then f maps Cauchy sequences into Cauchy sequences.

3. [2 marks] Let $\{f_n\}_{n\geq 1}$ be a sequence of functions which are continuous over $[0,1]$ and continuously differentiable in $(0,1)$. Assume that $|f_n(x)| \leq 1$ for all $x \in [0,1]$ and that $|f_n'(x)| \leq 1$ for all $x \in (0,1)$ and for each positive integer n. Pick out the true statements.
   (a) $\{f_n\}_{n\geq 1}$ is uniformly equicontinuous on $[0,1]$.
   (b) $\{f_n\}_{n\geq 1}$ is convergent uniformly on $[0,1]$.
   (c) $\{f_n\}_{n\geq 1}$ contains a subsequence which converges uniformly on $[0,1]$.

4. [3 marks] Let M be compact and let $f : M \to M$ satisfy $d(f(x), f(y)) = d(x,y)$ for all $x, y \in M$, Show that f is one-one and onto.

5. [3 marks] Let $\{f_n\}_{n\geq 1}$ be a sequence in $C[0,1]$ that converges uniformly on $[0,1]$. Prove that $\{f_n\}_{n\geq 1}$ is uniformly bounded.

6. [2+2 marks] Prove that the series
$$\sum_{n=1}^{\infty} ne^{-nx}$$
   converges uniformly on $[a,b]$ for $a, b > 0$. If $f(x) = \sum_{n=1}^{\infty} ne^{-nx}$, then show that $\int_{\log_e^2}^{\log_e^3} f(x)dx = \frac{1}{2}$.

7. [4 marks] Let g be a continuous function over $[0,1]$ and let $\{f_n\}_{n\geq 1}$ be a sequence of functions over $[0,1]$, where
$$f_n(x) = g(x)x^n \quad (x \in [0,1]).$$
   Prove that the sequence $\{f_n\}_{n\geq 1}$ converges uniformly on $[0,1]$ if and only if $g(1) = 0$.

8. [2+1+1 marks] Consider the normed linear space (NLS)

$$l_\infty := \{x = (x_1, x_2, \ldots,) : x_n \in \mathbb{R}, n \geq 1, x \text{ is a bounded sequence}\}$$

with the norm $\|x\|_\infty = \sup_{n\geq 1} |x_n|$. Let $K = \{x = (x_1, x_2, \ldots,) \in l_\infty : \lim_{n\to\infty} x_n = 1\}$. Prove that

(a) K is a closed subset of l_∞.

(b) If $T : l_\infty \to l_\infty$ is defined by $T(x) = (0, x_1, x_2, \ldots)$ for $x = (x_1, x_2, \ldots) \in l_\infty$, that is, if T shifts the entries forward and puts 0 in the empty slot, then $T(K) \subseteq K$.

(c) T is an isometry on K, but T has no fixed point in K.

9. (a) [3 marks] Define $T : C[a, b] \to C[a, b]$ by

$$(Tf)(x) = \int_a^x f(t)\, dt.$$

Show that T maps bounded sets into equicontinuous (and hence compact) sets.

(b) [3 marks] Let $\{f_n\}_{n\geq 1}$ be a sequence in $C[a, b]$ with $\|f_n\|_\infty \leq 1$ for all n, and define

$$F_n(x) = \int_a^x f_n(t)\, dt.$$

Show that some subsequence of $\{F_n\}_{n\geq 1}$ is uniformly convergent.